Recent advances on stochastic dynamic programming

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Workshop Stochastic Programming models and algorithms for energy planning

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Outline

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Motivation and goal.

- Consider a **multistage stochastic program** with $T_{\max}$ stages of form

$$\begin{array}{l} \inf \mathbb{E}_{\xi_2,\ldots,\xi_{T_{\max}}}\left[\sum_{t=1}^{T_{\max}} f_t(x_t, x_{t-1}, \xi_t)\right] \\ x_t \in X_t(x_{t-1}, \xi_t) \text{ a.s., } x_t\ \mathcal{F}_t\text{-measurable, } t = 1, \ldots, T_{\max}, \end{array}$$

where x_0 is given,

  - ξ_1 is deterministic, $(\xi_t)_{t=2}^{T_{\max}}$ is a stochastic process,
  - $\mathcal{F}_t$ is the sigma-algebra $\mathcal{F}_t := \sigma(\xi_j, j \le t)$, and
  - $X_t(x_{t-1}, \xi_t)$ is a subset of $\mathbb{R}^n$.

Motivation and goal

- For many applications modelled by multistage stochastic optimization problems, the optimization period is stochastic:
  - A company may want to determine optimal investments over its lifetime.
  - An individual may invest his money in financial assets until his death or until he obtains a given amount to cover an expense.
  - An hedge fund may have to deal with longetivity risk.
  - For capacity expansion, new electric power plants may be ready at random times.
- **Our goal:** define multistage stochastic optimization problems with a **random number of stages**, derive DP equations, and study solution methods.

- We assume that

  (H1) the number of stages T is a discrete random variable taking values in $\{2, \ldots, T_{\max}\}$.

- The number of stages T, or stopping time, induces the **Bernoulli process** D_t, $t = 1, \ldots, T_{\max}$ (a "Death" process), where $D_t = \mathbb{1}_{T>t}$ is the indicator of the event $\{T > t\}$:

$$D_t = \mathbb{1}_{T>t} = \begin{cases} 0 & \text{if the optimization period ended at } t \text{ or before,} \\ 1 & \text{otherwise.} \end{cases}$$

  Therefore T can be written as the following function of process (D_t):

$$T = \min \Big\{ 1 \leq t \leq T_{\max} : D_t = 0 \Big\}.$$

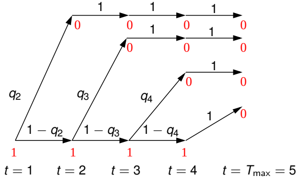
- Distribution of D_t given D_{t-1} is known as long as the distribution of T is known: if $p_t = \mathbb{P}(T = t)$ and $q_t = \mathbb{P}(D_t = 0 | D_{t-1} = 1)$, we have

$$q_t = \frac{p_t}{\prod_{k=2}^{t-1}(1 - q_k)}, t = 3, \ldots, T_{\max},$$

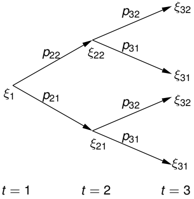
  and $q_2 = p_2$, $q_{T_{\max}} = 1$.

- We also have $\mathbb{P}(D_t = 0 | D_{t-1} = 0) = 1$ or equivalently $\mathbb{P}(D_t = 1 | D_{t-1} = 0) = 0$.

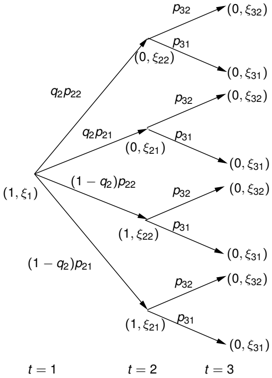
- There are at least two stages: D_1 takes value 1 with probability one.



Scenario tree for $(D_1, D_2, \ldots, D_{T_{\max}})$ with $T_{\max} = 5$



Scenario tree for $(\xi_1, \xi_2, \ldots, \xi_{T_{\max}})$ with $T_{\max} = 3$



Scenario tree for

$((D_1, \xi_1), (D_2, \xi_2), \ldots, (D_{T_{\max}}, \xi_{T_{\max}}))$ with $T_{\max} = 3$

Figure: Scenario trees (when ξ_t does not depend on $(\xi_{[t-1]}, D_t)$).

We come to the following definition of a **multistage stochastic optimization problem with a random number of stages**:

$$\begin{array}{l}\inf \mathbb{E}_{\xi_2,\ldots,\xi_{T_{\max}},D_2,\ldots,D_{T_{\max}}}\left[\sum_{t=1}^{T} f_t(x_t, x_{t-1}, \xi_t)\right]\\ x_t \in X_t(x_{t-1}, \xi_t) \text{ a.s., } x_t\ \overline{\mathcal{F}}_t\text{-measurable, } t = 1, \ldots, T_{\max},\end{array}$$

where $\overline{\mathcal{F}}_t$ is the sigma-algebra

$$\overline{\mathcal{F}}_t = \sigma(\xi_j, D_j, j \leq t).$$

Plugging relation $T = \min\left\{1 \leq t \leq T_{\max} : D_t = 0\right\}$ into the problem above we obtain the reformulation

$$\begin{array}{l}\inf \mathbb{E}_{\xi_2,\ldots,\xi_{T_{\max}},D_2,\ldots,D_{T_{\max}}}\left[\sum_{1\leq t\leq \min\{1\leq \tau\leq T_{\max}:D_\tau=0\}} f_t(x_t, x_{t-1}, \xi_t)\right]\\ x_t \in X_t(x_{t-1}, \xi_t) \text{ a.s., } x_t\ \overline{\mathcal{F}}_t\text{-measurable, } t = 1, \ldots, T_{\max}.\end{array}$$

Outline

State vector definition

The state vector for stage t contains:

- x_{t-1} (decision taken at the previous stage);
- the history $\xi_{[t-1]} = (\xi_1, \ldots, \xi_{t-1})$ of process (ξ_t) until stage $t-1$;
- past value D_{t-1} of (D_t).

  Other values $D_j, j \leq t-2$ of (D_t) are not necessary. Indeed:
  - if $D_{t-1} = 1$ then the whole history of (D_t) until $t-1$ is known: we know that $D_j = 1$ for $1 \leq j \leq t-1$;
  - on the other hand, if $D_{t-1} = 0$ then whatever the history of (D_t) until $t-1$, we know that the cost function is null for stage t because the optimization period ended at $t-1$ or before.

Consequently the **state vector** at stage t is $(x_{t-1}, \xi_{[t-1]}, D_{t-1})$ and we introduce for each stage $t = 2, \ldots, T_{\max}$, two functions:

- $\mathfrak{Q}_t$ such that $\mathfrak{Q}_t(x_{t-1}, \xi_{[t-1]}, D_{t-1}, \xi_t, D_t)$ is the optimal mean cost from t on starting at t from state $(x_{t-1}, \xi_{[t-1]}, D_{t-1})$ and knowing the values ξ_t and D_t of processes (ξ_t) and (D_t) at t;

- $\mathcal{Q}_t$ given by

$$\mathcal{Q}_t(x_{t-1}, \xi_{[t-1]}, D_{t-1}) = \mathbb{E}_{\xi_t, D_t}\Big[\mathfrak{Q}_t(x_{t-1}, \xi_{[t-1]}, D_{t-1}, \xi_t, D_t) | D_{t-1}, \xi_{[t-1]}\Big],$$

  i.e., $\mathcal{Q}_t(x_{t-1}, \xi_{[t-1]}, D_{t-1})$ is the optimal mean cost from t on starting at t from state $(x_{t-1}, \xi_{[t-1]}, D_{t-1})$.

We also set $\mathcal{Q}_{T_{\max}+1}(x_{T_{\max}}, \xi_{[T_{\max}]}, D_{T_{\max}}) \equiv 0$.

Dynamic Programming equations

For $t = 2, \ldots, T_{\max}$, we have

$$\mathcal{Q}_t(x_{t-1}, \xi_{[t-1]}, 0) = 0.$$

Next, for $t = 2, \ldots, T_{\max}$, functions $\mathcal{Q}_t(\cdot, \cdot, 1)$ satisfy the recurrence

$$\mathcal{Q}_t(x_{t-1}, \xi_{[t-1]}, 1) = \mathbb{E}_{\xi_t, D_t}\Big[\mathfrak{Q}_t(x_{t-1}, \xi_{[t-1]}, 1, \xi_t, D_t) | D_{t-1} = 1, \xi_{[t-1]}\Big]$$

where

$$\mathfrak{Q}_t(x_{t-1}, \xi_{[t-1]}, 1, \xi_t, 0) = \left\{ \begin{array}{l} \inf_{x_t} \;\; f_t(x_t, x_{t-1}, \xi_t) \\ x_t \in X_t(x_{t-1}, \xi_t), \end{array} \right.$$

and

$$\mathfrak{Q}_t(x_{t-1}, \xi_{[t-1]}, 1, \xi_t, 1) = \left\{ \begin{array}{l} \inf_{x_t} \;\; f_t(x_t, x_{t-1}, \xi_t) + \mathcal{Q}_{t+1}(x_t, \xi_{[t-1]}, \xi_t, 1) \\ x_t \in X_t(x_{t-1}, \xi_t). \end{array} \right.$$

DP equations can be written under the following compact form: for $t = 2, \ldots, T_{\max}$,

$$\mathcal{Q}_t(x_{t-1}, \xi_{[t-1]}, D_{t-1}) = \mathbb{E}_{\xi_t, D_t}\Big[\mathfrak{Q}_t(x_{t-1}, \xi_{[t-1]}, D_{t-1}, \xi_t, D_t) | D_{t-1}, \xi_{[t-1]}\Big] \quad (1)$$

where

$$\mathfrak{Q}_t(x_{t-1}, \xi_{[t-1]}, D_{t-1}, \xi_t, D_t) = \left\{ \begin{array}{l} \inf_{x_t} \; D_{t-1} f_t(x_t, x_{t-1}, \xi_t) + \mathcal{Q}_{t+1}(x_t, \xi_{[t-1]}, \xi_t, D_t) \\ x_t \in X_t(x_{t-1}, \xi_t). \end{array} \right. \quad (2)$$

Setting $D_0 = 1$, recalling that $D_1 = 1$, the optimal value of the original multistage problem can be expressed as

$$\inf_{x_1} \{D_0 f_1(x_1, x_0, \xi_1) + \mathcal{Q}_2(x_1, \xi_1, D_1) : x_1 \in X_1(x_0, \xi_1)\}$$

and therefore (1)-(2) are dynamic programming equations for

$$\begin{array}{l} \inf \mathbb{E}_{\xi_2, \ldots, \xi_{T_{\max}}, D_2, \ldots, D_{T_{\max}}} \Big[\sum_{t=1}^{T_{\max}} D_{t-1} f_t(x_t, x_{t-1}, \xi_t)\Big] \\ x_t \in X_t(x_{t-1}, \xi_t) \text{ a.s.}, \; x_t \; \overline{\mathcal{F}}_t\text{-measurable}, \; t = 1, \ldots, T_{\max}. \end{array} \quad (3)$$

Let us now consider the case when:

- ξ_t does not depend on $(\xi_{[t-1]}, D_t)$ and D_t only depends on D_{t-1}, i.e., (D_t) is an inhomogeneous Markov chain with two states: an absorbing state (when the optimization period is over) and a second state where the optimization period is still not over.
- The distribution of ξ_t is discrete with finite support $\{\xi_{t1}, \ldots, \xi_{tM_t}\}$ with $p_{tj} = \mathbb{P}(\xi_t = \xi_{tj})$.

With these assumptions, previous DP equations become:
$\mathcal{Q}_{T_{\max}+1}(x_{T_{\max}}, D_{T_{\max}}) \equiv 0$,

$$\mathcal{Q}_t(x_{t-1}, 0) = 0, \; t = 2, \ldots, T_{\max},$$

and recalling that $q_t = \mathbb{P}(D_t = 0 | D_{t-1} = 1)$, we have

$$\mathcal{Q}_t(x_{t-1}, 1) = (1 - q_t) \sum_{j=1}^{M_t} p_{tj} \mathfrak{Q}_t(x_{t-1}, 1, \xi_{tj}, 1) + q_t \sum_{j=1}^{M_t} p_{tj} \mathfrak{Q}_t(x_{t-1}, 1, \xi_{tj}, 0),$$

$$\mathfrak{Q}_t(x_{t-1}, 1, \xi_{tj}, 1) = \left\{ \begin{array}{l} \inf_{x_t} \;\; f_t(x_t, x_{t-1}, \xi_{tj}) + \mathcal{Q}_{t+1}(x_t, 1) \\ x_t \in X_t(x_{t-1}, \xi_{tj}), \end{array} \right.$$

and

$$\mathfrak{Q}_t(x_{t-1}, 1, \xi_{tj}, 0) = \left\{ \begin{array}{l} \inf_{x_t} \;\; f_t(x_t, x_{t-1}, \xi_{tj}) \\ x_t \in X_t(x_{t-1}, \xi_{tj}). \end{array} \right.$$

Outline

1. Dealing with a random number of stages
   - Problem formulation
   - Dynamic programming equations
   - Solution methods
2. Inexact variants of SDDP
   - Linear problems
   - Nonlinear problems
3. StoDCUP

- Recall DP equations when ξ_t does not depend on $(\xi_{[t-1]}, D_t)$ and D_t only depends on D_{t-1}, the distributions of T and ξ_t are discrete: the support of T is $\{2, \ldots, T_{\max}\}$ and the support of ξ_t is $\Theta_t = \{\xi_{t1}, \ldots, \xi_{tM_t}\}$ with $p_{ti} = \mathbb{P}(\xi_t = \xi_{ti}) > 0, i = 1, \ldots, M_t$.
- DP equations: for $t = 2, \ldots, T_{\max}$, we have

$$\mathcal{Q}_t(x_{t-1}, 0) = 0,\ t = 2, \ldots, T_{\max},$$

and

$$\mathcal{Q}_t(x_{t-1}, 1) = (1 - q_t) \sum_{j=1}^{M_t} p_{tj} \mathfrak{Q}_t(x_{t-1}, 1, \xi_{tj}, 1) + q_t \sum_{j=1}^{M_t} p_{tj} \mathfrak{Q}_t(x_{t-1}, 1, \xi_{tj}, 0),$$

where $q_t = \mathbb{P}(D_t = 0 | D_{t-1} = 1)$,

$$\mathfrak{Q}_t(x_{t-1}, 1, \xi_{tj}, 1) = \left\{ \begin{array}{l} \inf_{x_t}\ f_t(x_t, x_{t-1}, \xi_{tj}) + \mathcal{Q}_{t+1}(x_t, 1) \\ x_t \in X_t(x_{t-1}, \xi_{tj}), \end{array} \right.$$

and

$$\mathfrak{Q}_t(x_{t-1}, 1, \xi_{tj}, 0) = \left\{ \begin{array}{l} \inf_{x_t}\ f_t(x_t, x_{t-1}, \xi_{tj}) \\ x_t \in X_t(x_{t-1}, \xi_{tj}). \end{array} \right.$$

SDDP-TSto: adaptation of SDDP for MSP with T random

- Take

  $$X_t(x_{t-1}, \xi_t) = \{x_t \in \mathbb{R}^n : x_t \in \mathcal{X}_t,\ g_t(x_t, x_{t-1}, \xi_t) \leq 0,\ \ A_t x_t + B_t x_{t-1} = b_t\},$$

  and ξ_t contains in particular the random elements in matrices A_t, B_t, b_t.
- $\mathcal{Q}_t(\cdot, 1), t = 2, \ldots, T_{\max}$ are approximated by functions $\mathcal{Q}_t^k(\cdot, 1)$, $t = 2, \ldots, T_{\max}$, which are maximum of $k+1$ affine functions called cuts:

  $$\mathcal{Q}_t^k(x_{t-1}, 1) = \max_{0 \leq j \leq k} \theta_t^j + \langle \beta_t^j, x_{t-1} \rangle.$$

- **Sampling:** at iteration k, a realization of the number of stages T and a sample for $(\xi_t)_{1 \leq t \leq T_{\max}}$, are generated.
- **Forward pass of iteration k:** decisions $x_t^k, t = 1, \ldots, T_{\max}$, are computed on this sample in a forward pass replacing (unkown) function $\mathcal{Q}_t(x_{t-1}, 1)$ by $\mathcal{Q}_t^{k-1}(x_{t-1}, 1)$.
- **Backward pass of iteration k:** decisions x_t^k are used to compute coefficients $\theta_t^k, \beta_t^k, t = 2, \ldots, T_{\max}$.

- **SDDP-TSto, Step 1: Initialization.** For $t = 2, \ldots, T_{\max}$, take for $\mathcal{Q}_t^0(\cdot, 1)$ a known lower bounding affine function $\theta_t^0 + \langle \beta_t^0, \cdot \rangle$ for $\mathcal{Q}_t(\cdot, 1)$. Set $k = 1$ and $\mathcal{Q}_{T_{\max}+1}^0(\cdot, 1) = \mathcal{Q}_{T_{\max}+1}^0(\cdot, 0) \equiv 0$, $\mathcal{Q}_t^0(\cdot, 0) \equiv 0$, $t = 2, \ldots, T_{\max}$. Fix $0 < \text{Tol} < 1$ (for stopping criterion). Compute q_t, $t = 2, \ldots, T_{\max}$.
- **SDDP-TSto, Step 2: Forward pass.** We generate a sample

$$((\tilde{\xi}_1^k, \tilde{D}_1^k), (\tilde{\xi}_2^k, \tilde{D}_2^k), \ldots, (\tilde{\xi}_{T_{\max}}^k, \tilde{D}_{T_{\max}}^k)),$$

from the distribution of

$$\gamma^k = ((\xi_1^k, D_1^k), (\xi_2^k, D_2^k), \ldots, (\xi_{T_{\max}}^k, D_{T_{\max}}^k)) \sim ((\xi_1, D_1), (\xi_2, D_2), \ldots, (\xi_{T_{\max}}, D_{T_{\max}})),$$

with the convention that $\tilde{\xi}_1^k = \xi_1$, $\tilde{D}_1^k = 1$.
$\text{Cost}_k = 0$.
For $t = 1, \ldots, T_{\max}$, we compute an optimal solution x_t^k of

$$\left\{ \begin{array}{l} \inf_{x_t \in \mathbb{R}^n} \tilde{D}_{t-1}^k f_t(x_t, x_{t-1}^k, \tilde{\xi}_t^k) + \mathcal{Q}_{t+1}^{k-1}(x_t, \tilde{D}_t^k) \\ x_t \in X_t(x_{t-1}^k, \tilde{\xi}_t^k), \end{array} \right.$$

where $\tilde{D}_0^k = 1$ and $x_0^k = x_0$.
$\text{Cost}_k \leftarrow \text{Cost}_k + \tilde{D}_{t-1}^k f_t(x_t^k, x_{t-1}^k, \tilde{\xi}_t^k)$.
End For

Upper bound computation: If $k \geq N$ compute

$$\overline{\text{Cost}}_k = \frac{1}{N} \sum_{j=k-N+1}^{k} \text{Cost}_j, \quad \hat{\sigma}^2_{N,k} = \frac{1}{N} \sum_{j=k-N+1}^{k} [\text{Cost}_j - \overline{\text{Cost}}_k]^2$$

and the upper bound

$$\overline{U}_k = \overline{\text{Cost}}_k + \frac{\hat{\sigma}_{N,k}}{\sqrt{N}} t_{N-1,1-\alpha}$$

where $t_{N-1,1-\alpha}$ is the $(1-\alpha)$-quantile of the Student distribution with $N-1$ degrees of freedom.

- **SDDP-TSto, Step 3: Backward pass.** Let $\underline{\mathfrak{Q}}_t^k(x_{t-1}, D_{t-1}, \xi_t, D_t)$ be the function given by

$$\underline{\mathfrak{Q}}_t^k(x_{t-1}, D_{t-1}, \xi_t, D_t) = \left\{ \begin{array}{l} \inf_{x_t} \;\; D_{t-1} f_t(x_t, x_{t-1}, \xi_t) + \mathcal{Q}_{t+1}^k(x_t, D_t) \\ x_t \in X_t(x_{t-1}, \xi_t). \end{array} \right.$$

Set $\mathcal{Q}_{T_{\max}+1}^k(\cdot, 1) = \mathcal{Q}_{T_{\max}+1}^k(\cdot, 0) \equiv 0$.
For $t = T_{\max}$ down to $t = 2$,
Set $\mathcal{Q}_t^k(\cdot, 0) \equiv 0$.
For $j = 1, \ldots, M_t$,
Compute $\underline{\mathfrak{Q}}_t^k(x_{t-1}^k, 1, \xi_{tj}, 1)$, compute

$$\mathfrak{Q}_t(x_{t-1}^k, 1, \xi_{tj}, 0) = \left\{ \begin{array}{l} \inf_{x_t} \;\; f_t(x_t, x_{t-1}^k, \xi_{tj}) \\ x_t \in X_t(x_{t-1}^k, \xi_{tj}), \end{array} \right.$$

compute a subgradient β_{tj}^k of $\underline{\mathfrak{Q}}_t^k(\cdot, 1, \xi_{tj}, 1)$ at x_{t-1}^k and a subgradient γ_{tj}^k of $\mathfrak{Q}_t(\cdot, 1, \xi_{tj}, 0)$ at x_{t-1}^k.
End For

Compute

$$
\begin{aligned}
\theta_t^k &= (1-q_t)\sum_{j=1}^{M_t} p_{tj}\Big(\underline{\mathfrak{Q}}_t^k(x_{t-1}^k,1,\xi_{tj},1) - \langle \beta_{tj}^k, x_{t-1}^k\rangle\Big)\\
&\quad + q_t \sum_{j=1}^{M_t} p_{tj}\Big(\mathfrak{Q}_t(x_{t-1}^k,1,\xi_{tj},0) - \langle \gamma_{tj}^k, x_{t-1}^k\rangle\Big),\\
\beta_t^k &= (1-q_t)\sum_{j=1}^{M_t} p_{tj}\beta_{tj}^k + q_t \sum_{j=1}^{M_t} p_{tj}\gamma_{tj}^k.
\end{aligned}
$$

End For

Lower bound computation: compute the lower bound $\underline{L}_k$ on the optimal value given by

$$
\underline{L}_k = \begin{cases} \inf_{x_1} \; f_1(x_1,x_0,\xi_1) + \mathcal{Q}_2^k(x_1,1) \\ x_1 \in X_1(x_0,\xi_1). \end{cases}
$$

SDDP-TSto, Step 4: If $k \geq N$ and $\frac{\overline{U}_k - \underline{L}_k}{\overline{U}_k} \leq$ Tol then stop otherwise do $k \leftarrow k+1$ and go to Step 2.

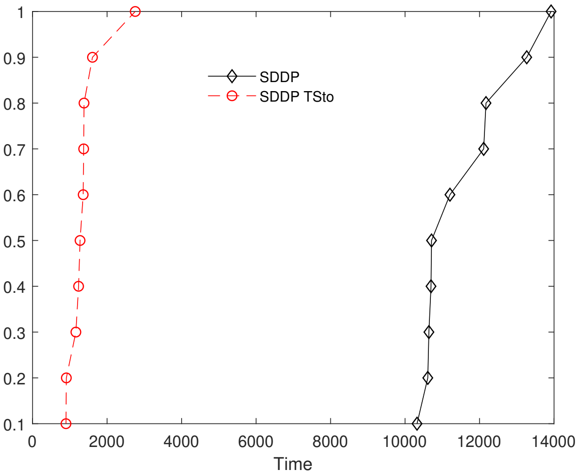


Figure: Computational time in seconds to solve an instance of a portfotlio problem with transaction costs with $T_{\max} = 10$, $n = 10$, $M = 100$ with SDDP and SDDP-TSto.

- Main idea of inexact SDDP: **solve approximately all subproblems** of forward and backward passes.
- Steps:
  - Derive **inexact cuts for value functions** of linear and convex nonlinear optimization problems (results on normal and tangeant cones and duality).
  - Show boundedness of approximate dual solutions.
  - **Convergence:** for bounded noises and vanishing noises. For vanishing noises, show that the distance between value functions and cuts at the trial points vanishes.

Outline

The first stage problem is

$$\mathcal{Q}_1(x_0) = \left\{ \begin{array}{l} \min_{x_1 \in \mathbb{R}^n} c_1^T x_1 + \mathcal{Q}_2(x_1) \\ A_1 x_1 + B_1 x_0 = b_1, x_1 \geq 0 \end{array} \right.$$

for x_0 given and for $t = 2, \ldots, T$, $\mathcal{Q}_t(x_{t-1}) = \mathbb{E}_{\xi_t}[\mathfrak{Q}_t(x_{t-1}, \xi_t)]$ with

$$\mathfrak{Q}_t(x_{t-1}, \xi_t) = \left\{ \begin{array}{l} \min_{x_t \in \mathbb{R}^n} {c_t}^T x_t + \mathcal{Q}_{t+1}(x_t) \\ A_t x_t + B_t x_{t-1} = b_t, x_t \geq 0, \end{array} \right.$$

with the convention that $\mathcal{Q}_{T+1}$ is null and where for $t = 2, \ldots, T$, random vector ξ_t corresponds to the concatenation of the elements in A_t, B_t, b_t, c_t.

- Inexact SDDP solves the DP equations computing at iteration k, δ_t^k-optimal solutions in the forward pass and ε_t^k-optimal solutions in the backward pass which are basic feasible solutions.

- We make the following assumptions:

(A0) (ξ_t) is interstage independent and for $t = 2, \ldots, T$, ξ_t is a random vector taking values in $\mathbb{R}^K$ with a discrete distribution and a finite support $\Theta_t = \{\xi_{t1}, \ldots, \xi_{tM}\}$ while ξ_1 is deterministic, with vector ξ_{tj} being the concatenation of the elements in $A_{tj}, B_{tj}, b_{tj}, c_{tj}$.

(A1-L) The set $X_1(x_0, \xi_1)$ is nonempty and bounded and for every $x_1 \in X_1(x_0, \xi_1)$, for every $t = 2, \ldots, T$, for every realization $\tilde{\xi}_2, \ldots, \tilde{\xi}_t$ of $\xi_2, \ldots, \xi_t$, for every $x_\tau \in X_\tau(x_{\tau-1}, \tilde{\xi}_\tau), \tau = 2, \ldots, t-1$, the set $X_t(x_{t-1}, \tilde{\xi}_t)$ is nonempty and bounded.

(A2) The samples in the backward passes are independent: $(\tilde{\xi}_2^k, \ldots, \tilde{\xi}_T^k)$ is a realization of $\xi^k = (\xi_2^k, \ldots, \xi_T^k) \sim (\xi_2, \ldots, \xi_T)$ and $\xi^1, \xi^2, \ldots,$ are independent.

Theorem (Convergence of ISDDP-LP with bounded errors)

Consider the sequences of decisions $(x_n^k)_{n\in\mathcal{N}}$ *and of functions* $(\mathcal{Q}_t^k)$ *generated by ISDDP-LP. Assume that (A0), (A1-L), and (A2) hold, and that errors* ε_t^k *and* δ_t^k *are bounded:* $0 \leq \varepsilon_t^k \leq \bar{\varepsilon}$, $0 \leq \delta_t^k \leq \bar{\delta}$ *for finite* $\bar{\delta}, \bar{\varepsilon}$*. Then the limit superior and limit inferior of the sequence* $\underline{\mathfrak{Q}}_1^{k-1}(x_0, \xi_1)$ *of lower bounds on the optimal value* $\mathcal{Q}_1(x_0)$ *satisfy almost surely*

$$\mathcal{Q}_1(x_0) - \bar{\delta}T - \bar{\varepsilon}(T-1) \leq \varliminf_{k\to+\infty} \underline{\mathfrak{Q}}_1^{k-1}(x_0, \xi_1) \leq \varlimsup_{k\to+\infty} \underline{\mathfrak{Q}}_1^{k-1}(x_0, \xi_1) \leq \mathcal{Q}_1(x_0).$$

Theorem (Convergence of ISDDP-LP with asymptotically vanishing errors)

Let Assumptions (A0), (A1-L), and (A2) hold. If for all $t = 1, \ldots, T$, $\lim\limits_{k\to+\infty} \delta_t^k = 0$ *and for all* $t = 1, \ldots, T-1$, $\lim_{k\to+\infty} \varepsilon_t^k = 0$*, then ISDDP-LP converges with probability one in a finite number of iterations to an optimal solution to the original problem.*

Outline

We consider problems of form

$$\inf_{x_1,\ldots,x_T} \mathbb{E}_{\xi_2,\ldots,\xi_T}[\sum_{t=1}^{T} f_t(x_t(\xi_1,\xi_2,\ldots,\xi_t), x_{t-1}(\xi_1,\xi_2,\ldots,\xi_{t-1}),\xi_t)]$$
$$x_t(\xi_1,\xi_2,\ldots,\xi_t) \in X_t(x_{t-1}(\xi_1,\xi_2,\ldots,\xi_{t-1}),\xi_t) \text{ a.s., } x_t\ \mathcal{F}_t\text{-measurable, } t \leq T,$$

where x_0 is given, $(\xi_t)_{t=2}^{T}$ is a stochastic process, $\mathcal{F}_t$ is the sigma-algebra $\mathcal{F}_t := \sigma(\xi_j, j \leq t)$, and where $X_t(x_{t-1}, \xi_t)$ is now given by

$$X_t(x_{t-1},\xi_t) = \{x_t \in \mathbb{R}^n : x_t \in \mathcal{X}_t,\ g_t(x_t, x_{t-1}, \xi_t) \leq 0,\ \ A_t x_t + B_t x_{t-1} = b_t\},$$

with ξ_t containing in particular the random elements in matrices A_t, B_t, and vector b_t.

Assumptions.

There exists $\varepsilon_t > 0$ such that for $t = 1, \ldots, T$,

- (A1-NL)-(a) $\mathcal{X}_t$ is nonempty, convex, and compact.
- (A1-NL)-(b) For every $j = 1, \ldots, M$, the function $f_t(\cdot, \cdot, \xi_{tj})$ is convex on $\mathcal{X}_t \times \mathcal{X}_{t-1}$ and belongs to $\mathcal{C}^1(\mathcal{X}_t \times \mathcal{X}_{t-1})$, the set of real-valued continuously differentiable functions on $\mathcal{X}_t \times \mathcal{X}_{t-1}$.
- (A1-NL)-(c) For every $j = 1, \ldots, M$, each component $g_{ti}(\cdot, \cdot, \xi_{tj}), i = 1, \ldots, p$, of function $g_t(\cdot, \cdot, \xi_{tj})$ is convex on $\mathcal{X}_t \times \mathcal{X}_{t-1}^{\varepsilon_t}$ and belongs to $\mathcal{C}^1(\mathcal{X}_t \times \mathcal{X}_{t-1})$ where $\mathcal{X}_{t-1}^{\varepsilon_t} = \mathcal{X}_{t-1} + \varepsilon_t \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$.
- (A1-NL)-(d) For every $j = 1, \ldots, M$, for every $x_{t-1} \in \mathcal{X}_{t-1}^{\varepsilon_t}$, the set $X_t(x_{t-1}, \xi_{tj}) \cap \text{ri}(\mathcal{X}_t)$ is nonempty.
- (A1-NL)-(e) If $t \geq 2$, for every $j = 1, \ldots, M$, there exists $\bar{x}_{tj} = (\bar{x}_{tjt}, \bar{x}_{tjt-1}) \in \text{ri}(\mathcal{X}_t) \times \mathcal{X}_{t-1}$ such that $g_t(\bar{x}_{tjt}, \bar{x}_{tjt-1}, \xi_{tj}) < 0$ and $A_{tj}\bar{x}_{tjt} + B_{tj}\bar{x}_{tjt-1} = b_{tj}$.

Main tools to extend the convergence analysis of SDDP to the convergence analysis of ISDDP:

- Derive formulas for **inexact cuts** for value functions of form

$$\mathcal{Q}(x) = \left\{ \begin{array}{l} \inf_{y \in \mathbb{R}^n} \ f(y,x) \\ y \in S(x) := \{y \in Y \ : \ Ay + Bx = b, \ g(y,x) \leq 0\}, \end{array} \right. \tag{4}$$

  and **control the accurcay** of these cuts;

- **Bounding ε-optimal dual solutions**;

- Show that the distance between the cuts and some lowerbounding function for the Bellman functions at the trial points goes to zero as $k \to +\infty$.

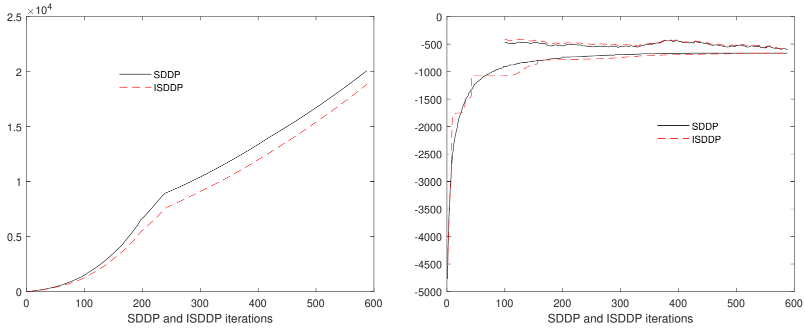
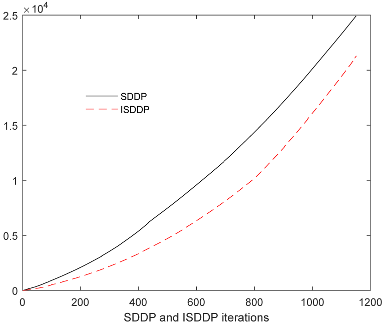


Figure: Portfolio problem: $M = 50$, $n = 50$, $T = 20$. Left: cumulative CPU time in seconds. Right: upper and lower bounds for ISDDP and SDDP.



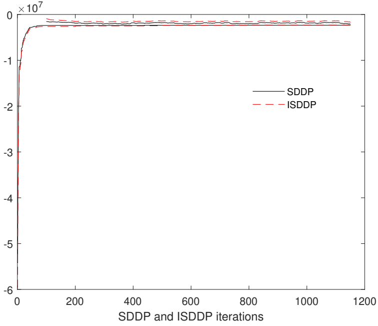


Figure: Portfolio problem: $M = 50$, $n = 10$, $T = 40$. Left: cumulative CPU time in seconds. Right: upper and lower bounds for ISDDP and SDDP.

StoDCuP: Stochastic Dynamic Cutting Plane.

- We present the ideas of the method using the deterministic equivalent of the method.
- Problem formulation: given $x_0 \in \mathcal{X}_0$, consider the optimization problem

$$\left\{ \begin{array}{l} \inf_{x_1,\ldots,x_T \in \mathbb{R}^n} \sum_{t=1}^{T} f_t(x_t, x_{t-1}) \\ g_t(x_t, x_{t-1}) \leq 0, \;\; A_t x_t + B_t x_{t-1} = b_t, \; t = 1, \ldots, T, \\ x_t \in \mathcal{X}_t, \; t = 1, \ldots, T, \end{array} \right. \tag{5}$$

We make the following assumptions:

(H1) There exists $\varepsilon_0 > 0$ such that for $t = 1, \ldots, T$:

(a) $\mathcal{X}_t \subset \mathbb{R}^n$ is nonempty, convex, and compact.

(b) f_t is a proper lower-semicontinuous convex function such that $\mathcal{X}_t \times \mathcal{X}_{t-1} \subset \text{int}(\text{dom} f_t)$.

(c) Each of the p components g_{ti}, $i = 1, \ldots, p$, of g_t is a proper lower-semicontinuous convex function such that $\mathcal{X}_t \times \mathcal{X}_{t-1} \subset \text{int}(\text{dom}\, g_{ti})$.

(d) For every $x_{t-1} \in \mathcal{X}_{t-1}^{\varepsilon_0}$, there exists $x_t \in \mathcal{X}_t$ such that $g_t(x_t, x_{t-1}) \leq 0$ and $A_t x_t + B_t x_{t-1} = b_t$.

(e) If $t \geq 2$, there exists $\bar{x}_t = (\bar{x}_{t,t}, \bar{x}_{t,t-1}) \in \text{ri}(\mathcal{X}_t) \times \mathcal{X}_{t-1}$ such that

$$A_t \bar{x}_{t,t} + B_t \bar{x}_{t,t-1} = b_t \text{ and } \bar{x}_t \in \text{ri}(\{g_t \leq 0\}).$$

DCuP Algorithm.

Step 0. Initialization. Let $\mathcal{Q}_t^0 : \mathcal{X}_{t-1} \to \mathbb{R}$, $t = 2, \ldots, T+1$, satisfying $\mathcal{Q}_t^0 \leq \mathcal{Q}_t$ be given and let $f_t^0, g_t^0 : \mathcal{X}_t \times \mathcal{X}_{t-1} \to \mathbb{R}$, $t = 1, \ldots, T$, be affine functions such that $f_t^0 \leq f_t, g_t^0 \leq g_t$. Set $k = 1$.

Step 1. Forward pass. Setting $x_0^k = x_0$, for $t = 1, 2, \ldots, T$, compute an optimal solution x_t^{2k-1} of

$$\left\{ \begin{array}{l} \min\limits_{x_t} \;\; f_t^{2k-2}(x_t, x_{t-1}^{2k-1}) + \mathcal{Q}_{t+1}^{k-1}(x_t) \\ x_t \in X_t^{2k-2}(x_{t-1}^{2k-1}), \end{array} \right.$$

where

$$X_t^k(x_{t-1}) = \{x_t \in \mathcal{X}_t : \; g_{ti}^k(x_t, x_{t-1}) \leq 0, i = 1, \ldots, p, \; A_t x_t + B_t x_{t-1} = b_t\},$$

Linearization step of the forward pass.

- Compute $f_t(x_t^{2k-1}, x_{t-1}^{2k-1})$ and $g_t(x_t^{2k-1}, x_{t-1}^{2k-1})$.
- Compute subgradients of f_t and g_t at $(x_t^{2k-1}, x_{t-1}^{2k-1})$.
- Compute linearizations $\ell_{f_t}(\cdot;(x_t^{2k-1}, x_{t-1}^{2k-1}))$ and $\ell_{g_t}(\cdot;(x_t^{2k-1}, x_{t-1}^{2k-1}))$ of f_t and g_t at $(x_t^{2k-1}, x_{t-1}^{2k-1})$.
- Define

$$\begin{aligned} f_t^{2k-1} &= \max\left(f_t^{2k-2}, \ell_{f_t}(\cdot;(x_t^{2k-1}, x_{t-1}^{2k-1}))\right), \\ g_t^{2k-1} &= \max\left(g_t^{2k-2}, \ell_{g_t}(\cdot;(x_t^{2k-1}, x_{t-1}^{2k-1}))\right). \end{aligned}$$

Step 2. Backward pass. For $t = T, T-1, \ldots, 2$, solve the problem

$$\underline{\mathcal{Q}}_t^k(x_{t-1}^{2k-1}) = \left\{ \begin{array}{l} \min_{x_t} \ f_t^{2k-1}(x_t, x_{t-1}^{2k-1}) + \mathcal{Q}_{t+1}^k(x_t) \\ x_t \in X_t^{2k-1}(x_{t-1}^{2k-1}). \end{array} \right. \tag{6}$$

Take a subgradient β_t^k of $\underline{\mathcal{Q}}_t^k(\cdot)$ at x_{t-1}^k, and store the new cut

$$\mathcal{C}_t^k(x_{t-1}) := \underline{\mathcal{Q}}_t^k(x_{t-1}^{2k-1}) + (\beta_t^k)^T(x_{t-1} - x_{t-1}^{2k-1})$$

for $\mathcal{Q}_t$, making up the new approximation $\mathcal{Q}_t^k = \max\{\mathcal{Q}_t^{k-1}, \mathcal{C}_t^k\}$.

Linearization step of the backward pass.
Denoting by x_t^{2k} an optimal solution of (6),

- compute $f_t(x_t^{2k}, x_{t-1}^{2k-1})$, $g_t(x_t^{2k}, x_{t-1}^{2k-1})$;
- compute subgradients of f_t and g_t at $(x_t^{2k}, x_{t-1}^{2k-1})$;
- compute linearizations $\ell_{f_t}(\cdot; (x_t^{2k}, x_{t-1}^{2k-1}))$ and $\ell_{g_t}(\cdot; (x_t^{2k}, x_{t-1}^{2k-1}))$ of f_t and g_t at $(x_t^{2k}, x_{t-1}^{2k-1})$.
- Define

$$f_t^{2k} = \max\left(f_t^{2k-1}, \ell_{f_t}(\cdot; (x_t^{2k}, x_{t-1}^{2k-1}))\right),$$
$$g_t^{2k} = \max\left(g_t^{2k-1}, \ell_{g_t}(\cdot; (x_t^{2k}, x_{t-1}^{2k-1}))\right).$$

Step 4. Do $k \leftarrow k+1$ and go to Step 1.

Theorem (Convergence analysis of DCuP)

Let Assumption (H1) holds. The limit of the sequence of upper bounds $\left(\sum_{t=1}^{T} f_t(x_t^{2k-1}, x_{t-1}^{2k-1})\right)_{k\geq 1}$ and of lower bounds $\underline{\mathcal{Q}}_1^{k-1}(x_0)$ is the optimal value $\mathcal{Q}_1(x_0)$ of (5) and any accumulation point of the sequence $(x_1^{2k-1}, \ldots, x_T^{2k-1})$ is an optimal solution to (5).

References

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